ON UPSTREAM BLOCKING IN A VISCOUS DIFFUSIVE STRATIFIED FLOW

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The effect of diffusion of specie upon the flow about a transverse flat plate moving horizontally in a viscous stratified medium is considered. Asymptotic expansions are used to define a parameter regime where a viscous-diffusive-buoyancy balance is dominant. The solution, expressed in terms of an inverse Fourier transform, is numerically integrated. The results show that, as in the non-diffusive problem, a region of closed streamlines exists ahead of the body. However, unlike the case where diffusion is neglected, the density field within this recirculating region is uniquely determined and found to be statically stable. It is also found that varying the relative amount of diffusion affects not only the density distribution, but the velocity profile as well, indicating a strong coupling between the vorticity and specie equation.

INTRODUCTION

It is a well-known feature of stratified fluid motions that the influence of a body moving horizontally can extend far in the upstream and downstream directions. In the limit of very large stratification and negligible viscous forces, Yih[1] has shown that this influence is contained in a slug of fluid of infinite horizontal extent. In reality, however, viscous and diffusive effects will modify this behavior so that far from the body the fluid is undisturbed. Long[2] has shown that, far upstream of an obstacle, the velocity perturbation decays algebraically as $X^{-3/4}$ as a result of viscous stresses. In a subsequent work, Long [3] showed that diffusion of the stratifying agent across streamlines alters this far field behavior such that the velocity decays as $X^{-2/3}$. This $X^{-2/3}$ dependance was also obtained by Koh[4] for the case of a viscous-diffusive flow towards a sink.

Graebel[5], using Fourier integral techniques, extended Long's[2] far field similarity analysis and found a solution for the upstream influence of a body in a nondiffusive fluid which is uniformly valid for $X \geqslant 0(R_e R_i)$, where R_e and R_i are the Reynolds number and Richardson number respectively. Graebel[5] showed that this solution is valid provided $R_i \gg 1$, $R_e R_i \gg 1$, and the Prandtl number is infinite. However, Browand and Winant[6], in discussing Graebel's[5] analysis, showed that this solution is uniformly valid for the entire flow field, except in the neighborhood of the singular points at the edges of the body, provided that the additional requirement $\sqrt{R_e/R_i} \ll 1$ is satisfied. The calculations of Browand and Winant[6] show that immediately ahead of the body there is a large recirculation

region containing closed streamlines, and they experimentally verified the existence of such a region. However, because diffusion of the stratifying agent was neglected, a unique solution for the density distribution within this region containing closed streamlines does not exist. Thus, Browand and Winant[6] were unable to present a realistic description of the density structure within this region.

The present study extends Graebel's analysis to include the diffusion term in the energy equation. The results show that when the stratifying agent is allowed to diffuse across streamlines, the density structure within the recirculating region is uniquely determined, and the calculations reveal a density structure which is physically plausible. The method employs asymptotic expansions applied in a straightforward manner, and it is shown that the elegant procedure followed by Freund and Meyer[7] who also considered this problem, is not required.

ANALYSIS

The problem to be considered is a linearly stratified infinite fluid medium flowing past a two dimension transverse flat plate, as depicted in Fig. 1.

For a steady two dimensional flow of an incompressible Boussinesq fluid, the equations governing the vorticity and energy are given by:

$$\left[\frac{1}{R_i} L\ (x,\ z,\Psi) - \frac{1}{R_i R_e} \nabla^2\right] \nabla^2\Psi - \frac{\partial \rho'}{\partial z} = 0 \qquad (1)$$

$$\left[L\ (x,\ z,\Psi) - \frac{1}{R_e S_c} \nabla^2\right] \rho' + \frac{\partial \Psi}{\partial x} = 0 \qquad (2)$$

where,

$$L\ (x,z,\psi) = \psi_z \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial z}$$

$$\rho\,(x,z) = (\rho_o - \beta z) + \rho'(x,z)$$

$$R_i = \frac{gh^2\beta}{U_o^{\,2}\rho_o}, \quad R_e = \frac{U_o h}{\nu}, \quad S_c = \frac{\nu}{D}, \quad u = \psi_z, \quad w = -\psi_x$$

Coordinates are nondimensionalized by h, velocities by U_o, and ν and D are respectively the kinematic viscosity and diffusion coefficient.

The far field asymptotic behavior of Eqs. 1 and 2 may be examined by compressing the horizontal coordinate by $\gamma(R_e, R_i, S_c)$ such that $\bar{x} = x\gamma$ ($\gamma \ll 1$), and scaling the density perturbation by $\delta(R_e, R_i, S_c)$ such that $\rho'(x,z) = \delta\bar{\rho}'(\bar{x},z)$ ($\delta \ll 1$). By introducing this scaling into the governing equations it is found that a viscous-buoyancy balance exists in the far field provided

$$\gamma = \frac{1}{R_e\sqrt{R_i S_c}} \ll 1$$

$$\delta \doteq \sqrt{\frac{S_c}{R_i}}$$

It is also found that the nonlinear terms in the vorticity and energy equation may be neglected provided

$$1/\sqrt{R_i S_c} \ll 1$$

$$\sqrt{\frac{S_c}{R_i}} \ll 1$$

The equations, to this order, may be combined to yield

$$\frac{\partial^6 \bar{\varsigma}'}{\partial z^6} + \frac{\partial^2 \bar{\varsigma}'}{\partial \bar{x}^2} = 0 \tag{3}$$

Eq. 3 may be solved using Fourier transform techniques. By imposing a boundedness constraint upon $\bar{\varsigma}'$ as $\bar{x} \rightarrow \pm \infty$, the solution may be written as:

$$\bar{\varsigma}'(\bar{x},z) = \int_0^{\infty} F(k) e^{-k^3 \bar{x}} \sin(kz)\,dk \tag{4}$$

where F(k) is the Fourier sine transform of the "initial condition" at x = 0. The derivation of this quantity is the crucial element in the solution of the present problem (and also in the solution presented by Graebel[5]), but the physical reasoning entering its determination has not here-to-fore been made explicit. Hence, we present a discussion of the intricacies involved in both the Graebel[5] and Freund and Meyer[7] solutions.

The important elements in specifying the "initial condition" are that the horizontal mass flux is conserved and that the integral solution (4), when written in terms of Long's[3] upstream wake similarity solution (which must hold as $x \rightarrow \infty$), matches uniformly with that solution. That is, the correct initial condition must yield a solution satisfying the constraint of constant momentum flux defect in the upstream wake and continuity of horizontal mass flux at x = 0. This is sketched illustratively in Fig. 2. The only conditions meeting these two requirements simultaneously are those with

$$u(x = 0, z) = \begin{Bmatrix} -1 & , |z| > 1 \\ -\left[\delta(z+1) + \delta(z-1)\right], & |z| \leq 1 \end{Bmatrix} \tag{5a}$$

for nondiffusive motion and

$$u(x = 0, z) = \begin{Bmatrix} 0 & , |z| < 1 \\ -1 - (z^2-1)^{-\frac{1}{2}} \left[|z| + (z^2 - 1)^{\frac{1}{2}}\right]^{-1} & , |z| \geq 1 \end{Bmatrix} \tag{5b}$$

for the diffusive case. The first condition was used without comment by Graebel[5], and the latter condition was derived via an integral equation approach by Fruend and Meyer[7]. Note that the effect of diffusion is clearly seen in spreading the initial condition and weakening the singularities at $z = \pm 1$. The strength of these (integrable) singularities are determined solely by the mass flux requirement, and their type is determined by the asymptotic momentum flux requirement.

Thus, by imposing upon Eq. 4 the condition at $\bar{x} = 0$ suggested by Fruend and Meyer[7] and given by Eq. 5b, we obtain their solution which is written as:

$$\begin{Bmatrix} u \\ w \\ \bar{\rho}' \\ p \end{Bmatrix} (\bar{x}, z) = \int_0^\infty J_1(k) e^{-k^3 \bar{x}} \begin{Bmatrix} \cos(kz) \\ k^2 \sin(kz) \\ \sin(kz) \\ k^{-1} \cos(kz) \\ k^{-1} \sin(kz) \end{Bmatrix} dk \quad (6)$$

The solution given by Eq. 6 is valid within the region $x = 0\,(R_e\sqrt{R_i S_c})$. Close to the plate, the equations may be rescaled in shear layer coordinates (see Fig. 3). Within this region a second solution having a viscous-buoyancy balance may be obtained by stretching the vertical coordinate as:

$$\bar{z} = \frac{(z-1)}{\varepsilon} = \frac{z-1}{\left[\dfrac{1}{R_e\sqrt{R_i S_c}}\right]^{1/3}}$$

Nonlinearity may be neglected provided

$$\left[\frac{R_e}{R_i S_c}\right]^{1/3} << 1$$

$$\left[\frac{R_e S_c^2}{R_i}\right]^{1/3} << 1$$

and the resulting equation is identical to Eq. 3, which governs the far field flow. However, contrary to the nondiffusive problem (Browand and Winant[6]), no similarity solution is possible in this case. Thus, under these more stringent conditions, the entire flow field, except in the neighborhood of the singular points at $z = \pm 1$, is governed by a linear viscous-buoyancy balance, and the solution given by Eq. 6 is uniformly valid. The importance of the inertial terms near the edges of the plate shows clearly the need to derive the appropriate "initial condition" via the procedure discussed above - a procedure which circumvents solution of the complete equations in the neighborhood of these singular points with subsequent asymptotic matching to determine $F(k)$.

RESULTS

The solution given by Eq. 6 was numerically integrated and the results are given in Figs. 4-8. Figure 4 shows the streamline pattern surrounding the body, and it is observed that the recirculation region ahead (and symmetrically behind) the body exists even when diffusion across streamlines is not negligible. Although not shown in Fig. 4, it is noted that, as in the case considered by Browand and Winant[6], an infinite number of such recirculating regions exist. It should be remembered, however, that the present solution is uniquely determined within these regions, while theirs is not.

Figure 5 shows the horizontal velocity ahead of the body in the plane of symmetry. The stagnation point located furthest upstream occurs at $\bar{x}^{-1/3} = .335$, and this defines the extent of the recirculating fluid. Also shown in this figure is Long's[3] similarity solution which satisfactorally

approximates the flow field for $\bar{x}^{1/3} > 1.4$.

Figure 6 shows the velocity distribution at various upstream locations both within and outside of the region of closed streamlines. One can see the marked increase in velocity as the singular points located at the edges of the plate are approached.

It is interesting to note that the velocity profile contains a single inflection point around $z = 1$ and then monotonically decays toward zero. This is contrasted with the oscillatory decay of the velocity at large values of z for the nondiffusive problem

The density perturbation profiles at various horizontal locations are shown in Fig. 7. It should be remembered that the density perturbation is scaled by the parameter $\sqrt{S_c/R_i}$ which must be small if the theory is to be valid. Thus, the density profile through the recirculation region is always statically stable, and varies only slightly from the linear gradient of the unperturbed fluid. This shows that, for the present model, the velocities are so low and diffusion is so important that, as a fluid parcel is convected about, it is never far from being neutrally buoyant.

The effect of different diffusion rates upon the velocity and density profiles is shown in Fig. 8. The two cases shown are $S_c = .72$ and 6.5, which roughly correspond to thermal diffusion in air and water respectively. It is interesting to note that changing the Schmidt number not only affects the density distribution, but also has a profound influence upon the velocity profile. This indicates a strong coupling between the vorticity equation and the energy equation.

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FIGURE CAPTIONS

FIG. 1. Geometry and coordinate system.

FIG. 2. Required horizontal velocity at $x = 0$ for diffusive and nondiffusive case.

FIG. 3. Shear layer coordinate system close to the singular points at the edge of the plate.

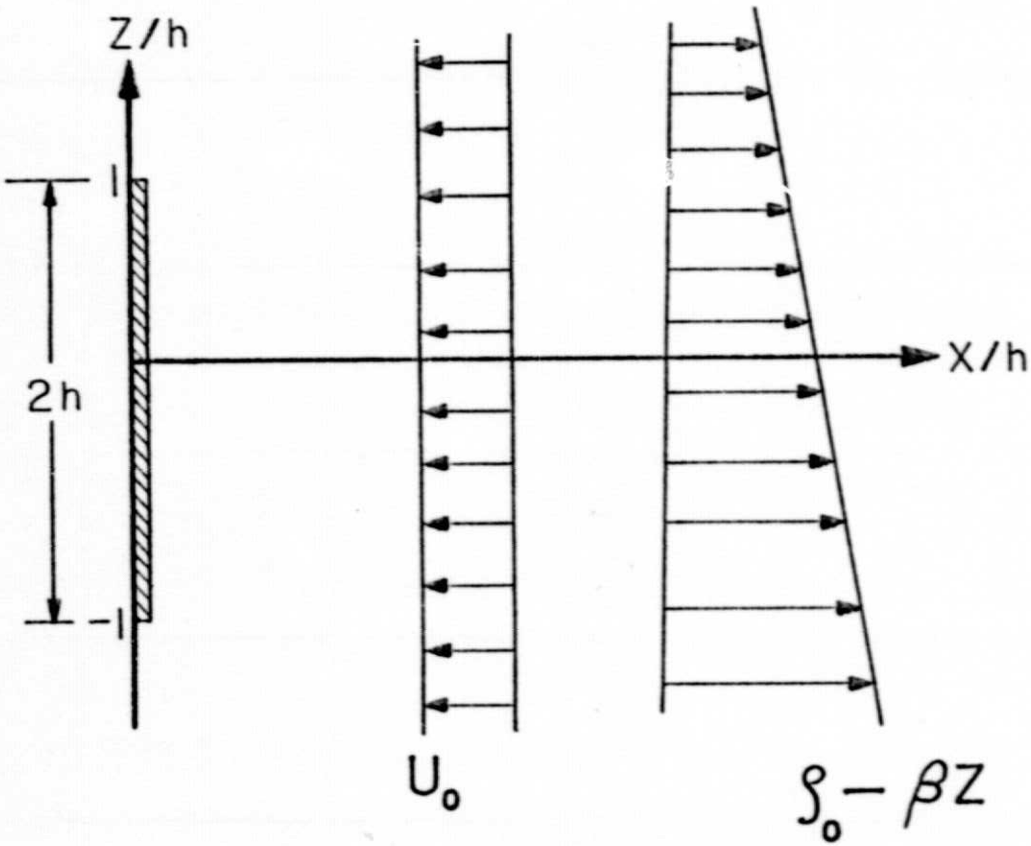
FIG. 4. Stream line pattern showing the recirculating region.

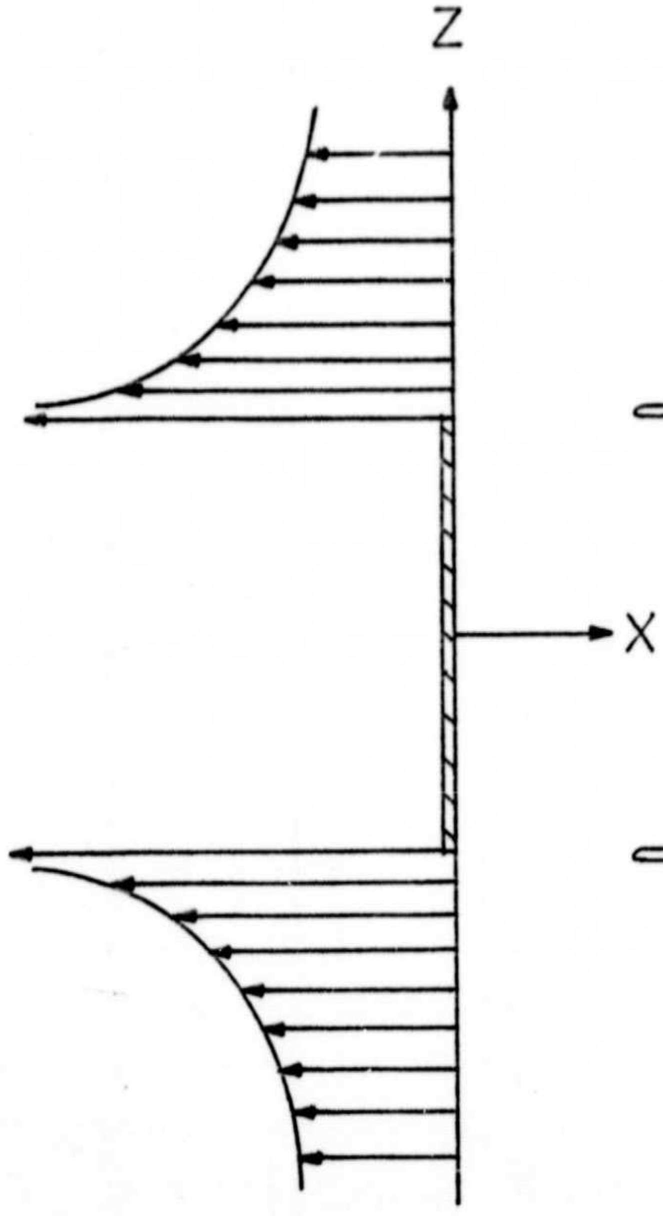
FIG. 5. Upstream variation of centerline horizontal velocity.

FIG. 6. Horizontal velocity profile at various upstream locations.

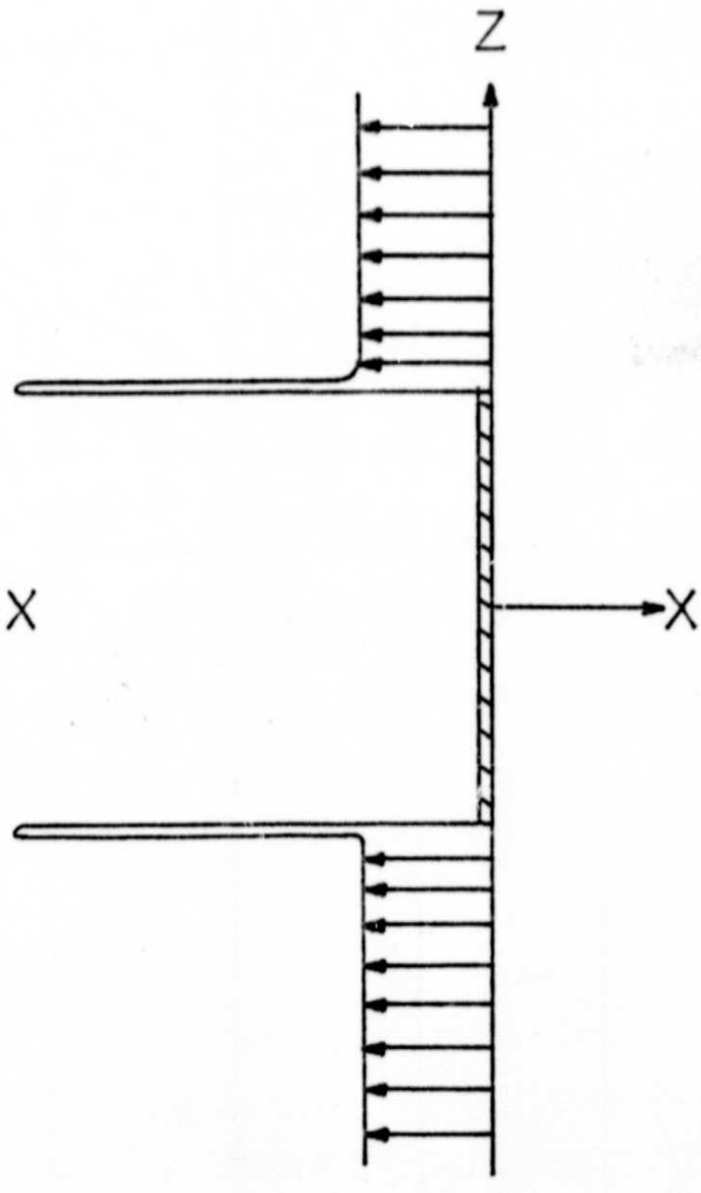
FIG. 7. Density perturbation profiles at various upstream locations.

FIG. 8. Effect of diffusion upon density perturbation and horizontal velocity profiles.

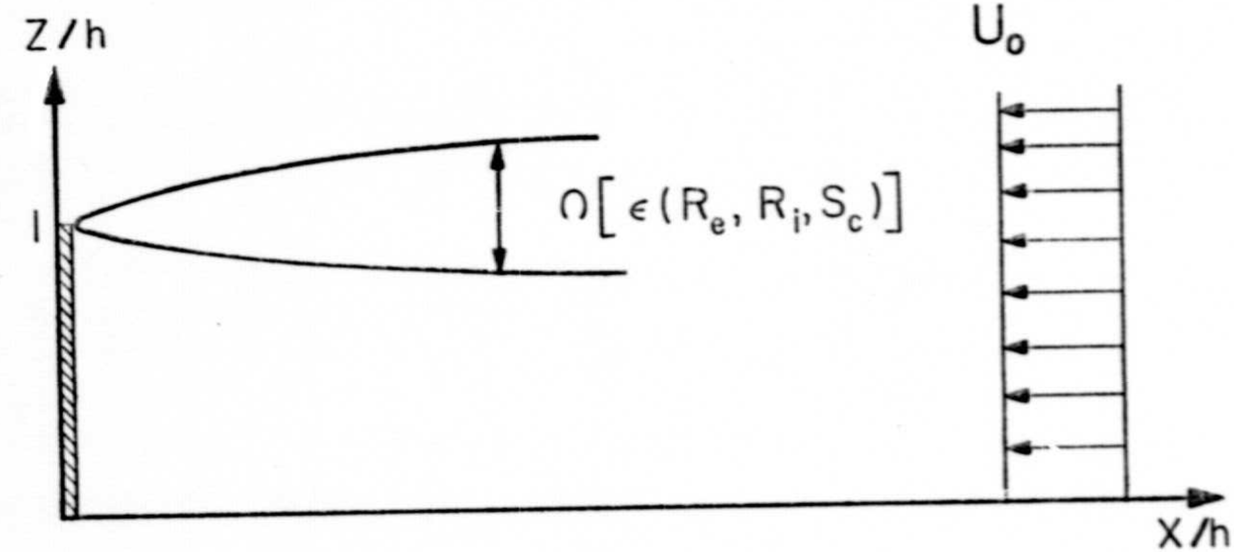

Z/h
1
-1
2h
X/h
U_o
$\rho_o - \beta Z$

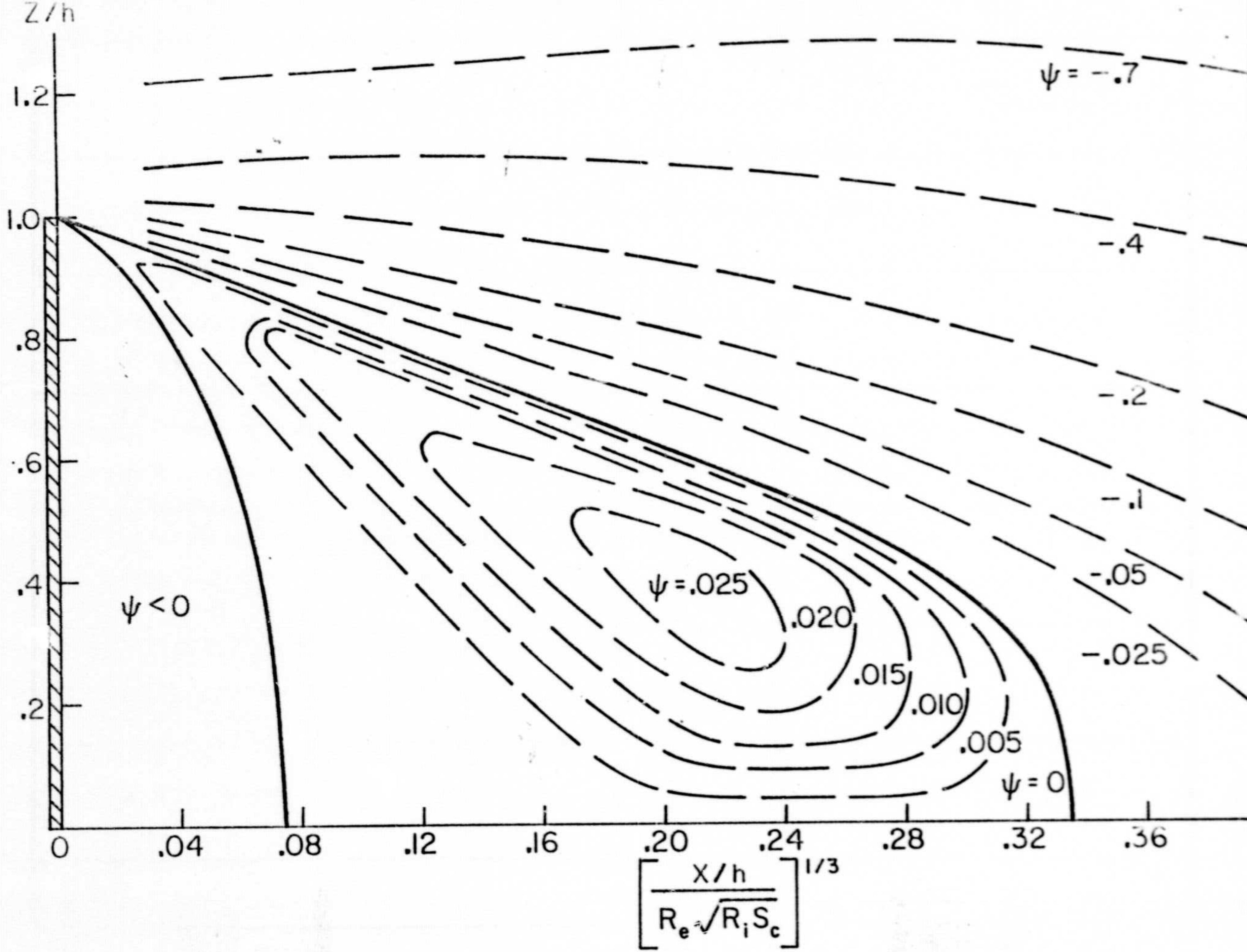


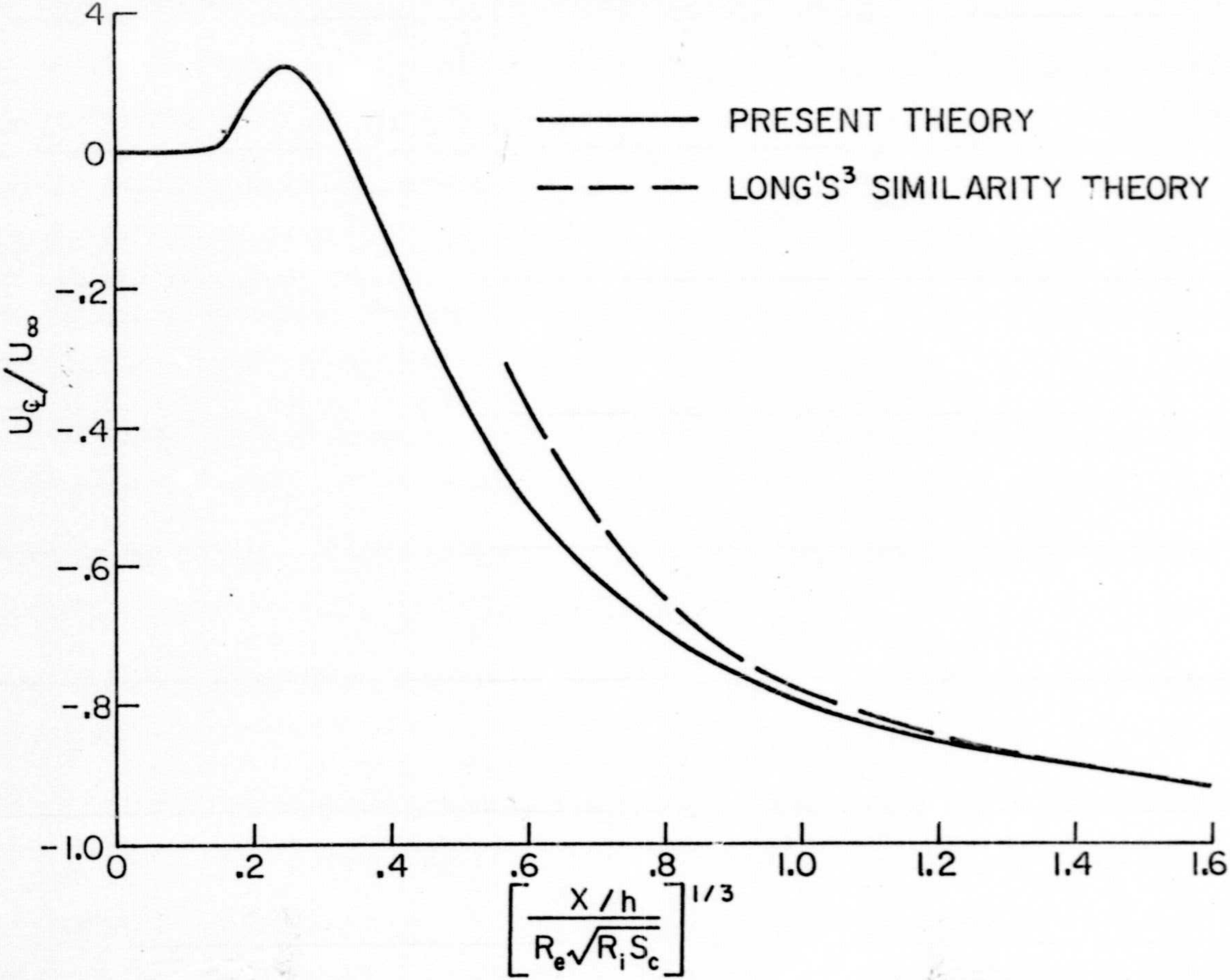
DIFFUSIVE CASE



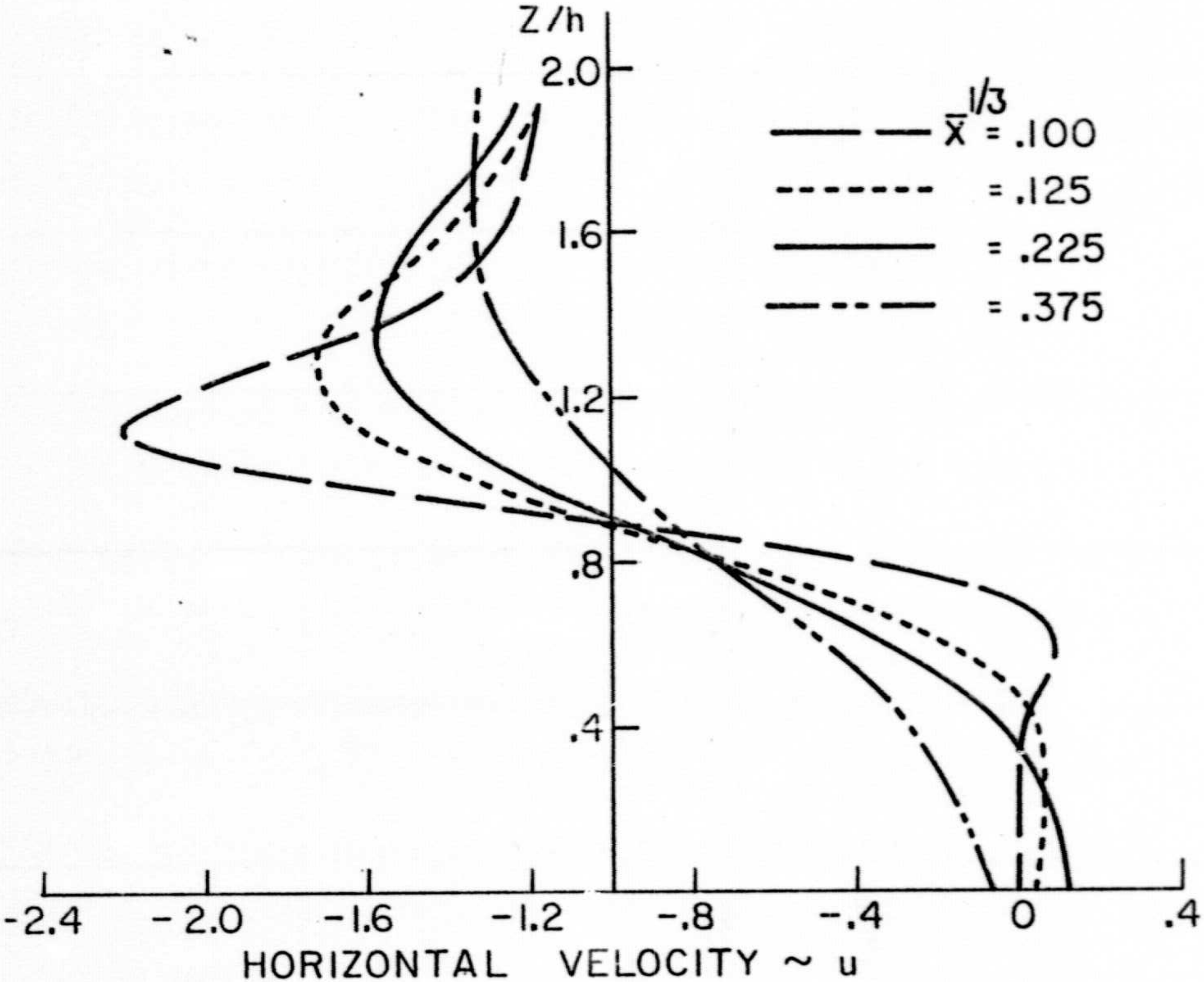
NONDIFFUSIVE CASE

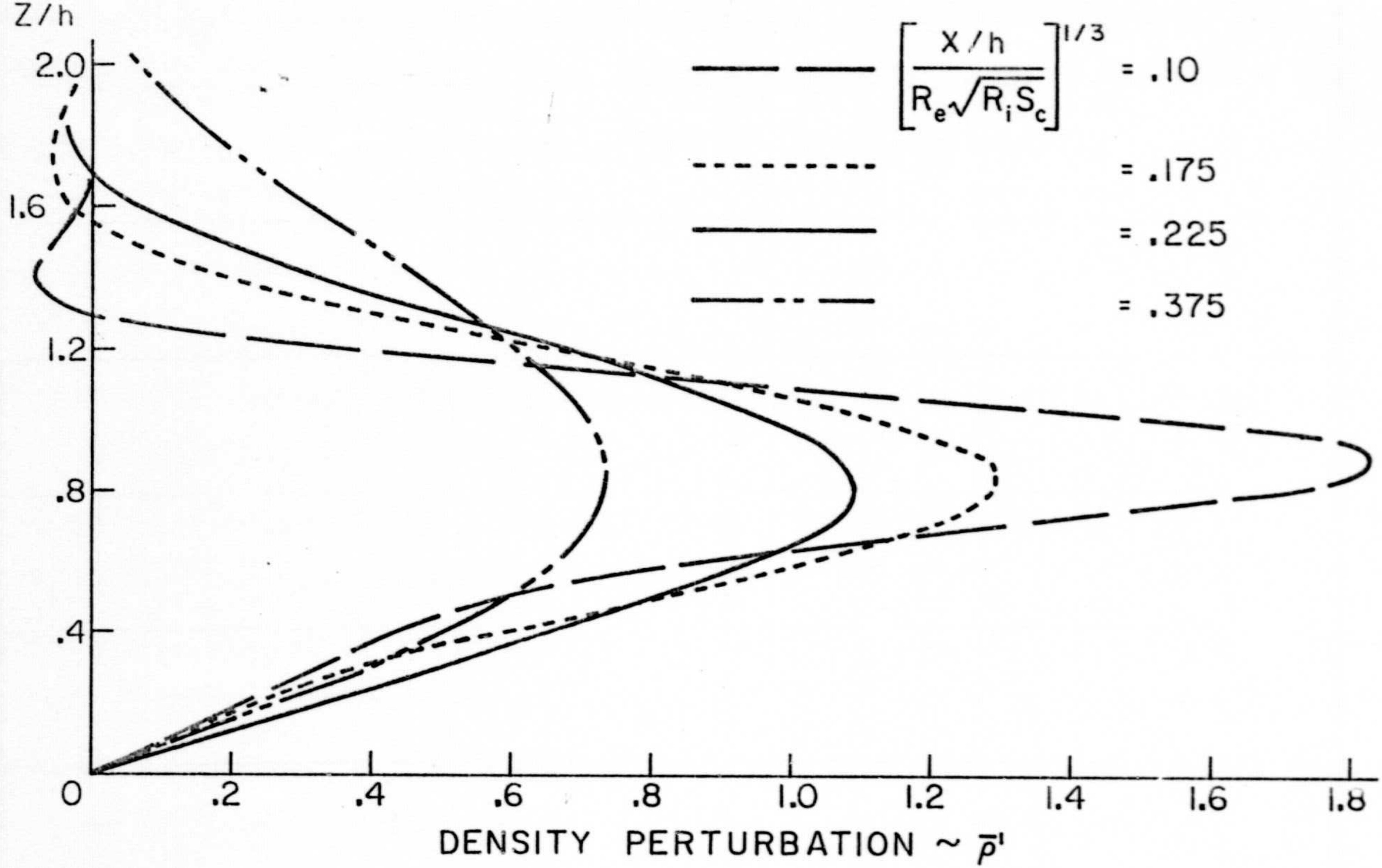

Z / h
U₀
1
O[ε(Re, Ri, Sc)]
X / h


Z/h
1.2
1.0
.8
.6
.4
.2
0
.04
.08
.12
.16
.20
.24
.28
.32
.36
$\left[\frac{X/h}{R_e\sqrt{R_i S_c}}\right]^{1/3}$
$\psi = -.7$
$-.4$
$-.2$
$-.1$
$-.05$
$-.025$
$\psi < 0$
$\psi = .025$
.020
.015
.010
.005
$\psi = 0$

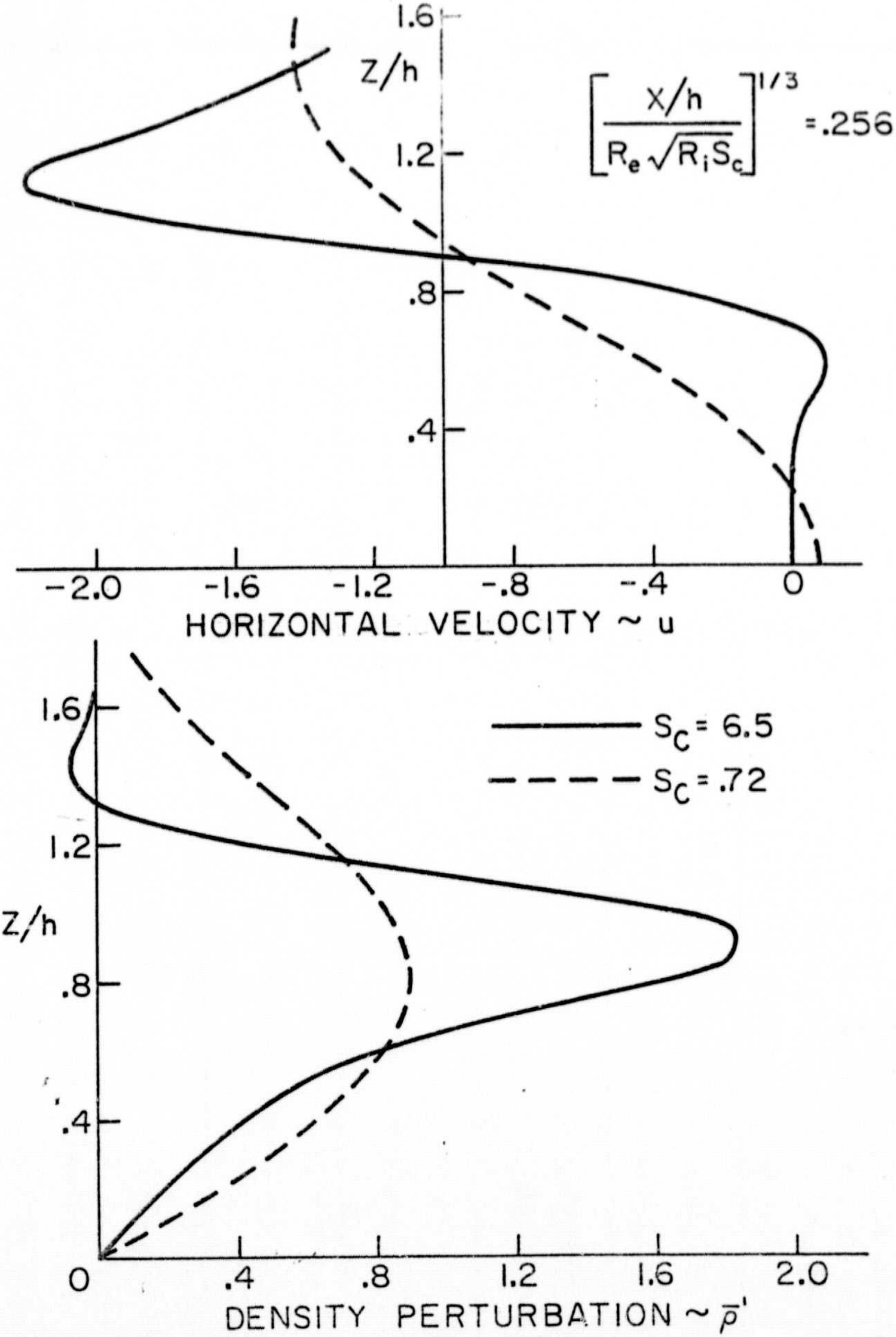


PRESENT THEORY
LONG'S[3] SIMILARITY THEORY
$U_{\mathbb{C}}/U_\infty$
4
0
-.2
-.4
-.6
-.8
-1.0
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
$\left[\frac{X/h}{R_e\sqrt{R_i S_c}}\right]^{1/3}$


Z/h
2.0
1.6
1.2
.8
.4
$\bar{X}^{1/3}$ = .100
= .125
= .225
= .375
-2.4
-2.0
-1.6
-1.2
-.8
-.4
0
.4
HORIZONTAL VELOCITY ~ u



Z/h
2.0
1.6
1.2
.8
.4
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
DENSITY PERTURBATION ~ $\bar{\rho}'$
$\left[\frac{X/h}{R_e\sqrt{R_i S_c}}\right]^{1/3} = .10$
$= .175$
$= .225$
$= .375$



1.6
Z/h
1.2
.8
.4
$\left[\frac{X/h}{R_e\sqrt{R_i S_c}}\right]^{1/3} = .256$
−2.0
−1.6
−1.2
−.8
−.4
0
HORIZONTAL VELOCITY ~ u
1.6
1.2
Z/h
.8
.4
$S_C = 6.5$
$S_C = .72$
0
.4
.8
1.2
1.6
2.0
DENSITY PERTURBATION ~ $\bar{\rho}'$